to higher accuracy without the addition of more complexity in the formulas used. We have shown how to convert the NORAD element set into epicycle parameters used in the analytic epicycle formulation of an orbit. This formulation is accurate to 10^{-7} and can be extended to higher levels of accuracy if required. We have shown that the correspondence between the NORAD elements and the epicycle parameters provides high levels of accuracy in satellite prediction over a timescale of 7 days.

As a result of this work, it is now possible to use a more accurate analytic model of satellite orbits and have access to orbital parameters through the widespread availability of NORAD data sets. The analytic models can be used on the ground for mission analysis tasks, Internet access to satellite data, and onboard satellites for autonomous operation.

References

[1]Hashida, Y., and Palmer, P., "Epicyclic Motion of Satellites about an Oblate Planet," *Journal of Guidance, Control, and Dynamics* (to be published).

[2]Hoots, F. R., and Roehrich, R. L., "Models for Propagation of NORAD Element Sets," SPACETRACK Rept. 3., Aerospace Defence Center, Peterson AFB, Colorado Springs, CO, 1980, pp. 10–20.

[3]Escobal, P., *Methods of Orbit Determination*, Wiley, New York, 1976, pp. 162–174.

[4]Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Drag," *Astronomical Journal*, Vol. 64 , No. 1274, 1959, pp. 378–397.

Closed-Form Solution of Line-of-Sight Trajectory for Maneuvering Target

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I. Introduction

IN previous studies[1–4] the differential equation

$$\ddot{\boldsymbol{r}} + f\dot{\boldsymbol{r}} + g\boldsymbol{r} = 0 \tag{1}$$

was used to describe planar elliptical trajectories under a noncentral force field, where f and g are scalar functions of the polar variables r and θ and/or their derivatives. It was shown that for planar motion

$$f = -(\dot{l}/l) \tag{2}$$

where $l = r^2\dot{\theta}$ is the angular momentum and dots indicate differentiation with respect to time. The application of Eq. (1) is extended here to the study of the trajectories of a homing missile under various guidance laws. It is shown that scattered published results in the literature can all be derived from Eq. (1). Moreover, a trajectory equation is derived for a homing missile in the case where the target velocity and the missile velocity are varying.

II. Guidance Law

Equation (1) can be easily reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\boldsymbol{r}}}{l}\right) = -\frac{g}{l}\boldsymbol{r} \tag{3}$$

where $\boldsymbol{r}$ is the vector corresponding to the line-of-sight distance $\boldsymbol{r} = \boldsymbol{r}_m - \boldsymbol{r}_t$, $\boldsymbol{r}_m$ is the missile distance, and $\boldsymbol{r}_t$ is the target distance from the origin (Fig. 1). Equation (3) can be written in components form as

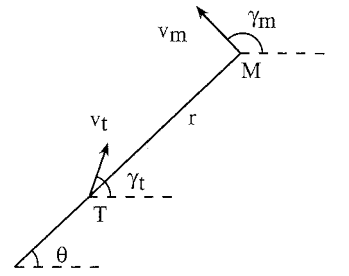

Fig. 1 Scenario for a missile-target interception.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_r}{l}\right) - \frac{\dot{\theta}}{l}v_\theta = -\frac{g}{l}r \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_\theta}{l}\right) + \frac{\dot{\theta}}{l}v_r = 0 \tag{5}$$

where $v_r = \dot{r}$ is the radial component of the velocity and $v_\theta = r\dot{\theta}$ is the transverse component of the velocity. In the case where l is constant, Eqs. (4) and (5) reduce to the case corresponding to the closed-form solution of generalized proportional navigation treated by Yang et al.[5] If the pursuer is always on the line of sight, then a collision will result if the polar angles of the missile and target are such that $\theta_m(t) = \theta_t(t) = \theta(t)$. Equation (3) can also be written in a form giving the radial and transverse components of the acceleration

$$\ddot{r} - r\dot{\theta}^2 = -gr + (\dot{l}/l)\dot{r} \tag{6}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = (\dot{l}/l)r\dot{\theta} \tag{7}$$

By differentiating $x = r\cos\theta$ and $y = r\sin\theta$, one obtains the x component $\dot{x} = v\cos\gamma$ and the y component $\dot{y} = v\cos\gamma$ of the velocity vector $\dot{\boldsymbol{r}}$. They can be written as

$$v\cos\gamma = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{8}$$

$$v\sin\gamma = \dot{r}\cos\theta + r\dot{\theta}\cos\theta \tag{9}$$

with

$$\tan\gamma = \frac{\dot{r}\sin\theta + r\dot{\theta}\cos\theta}{\dot{r}\cos\theta - r\dot{\theta}\sin\theta} \tag{10}$$

where γ is the angle between the direction of the velocity v and the x axis. By differentiating Eq. (10) with respect to θ, one obtains the following equation:

$$\frac{1}{\cos^2\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = \frac{r^2 + 2(\mathrm{d}r/\mathrm{d}\theta)^2 - r(\mathrm{d}^2r/\mathrm{d}\theta^2)}{[(\mathrm{d}r/\mathrm{d}\theta)\cos\theta - r\sin\theta]^2} \tag{11}$$

By noting that

$$\frac{1}{\cos^2\gamma} = \frac{(\mathrm{d}r/\mathrm{d}\theta)^2 + r^2}{[(\mathrm{d}r/\mathrm{d}\theta)\cos\theta - r\sin\theta]^2} \tag{12}$$

we finally get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = 2 - \frac{r^2 + r(\mathrm{d}^2r/\mathrm{d}\theta^2)}{r^2 + (\mathrm{d}r/\mathrm{d}\theta)^2} \tag{13}$$

When Eqs. (6) and (7) are used, Eq. (13) yields

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = \frac{g}{\dot{\theta}^2}\sin^2\psi \tag{14}$$

with

$$\sin^2\psi = r^2 \bigg/ \left[r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right] \tag{15}$$

where $\psi = \gamma - \theta$ is the angle between v and r. Equation (14) gives a guidance law between γ and θ (or ψ and θ) corresponding to the

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trajectory described by the differential equation given by Eq. (1). Note that, in the derivation of Eqs. (14) and (15), no restriction is placed on the time variation of the velocities v_t and v_m of target and missile.

III. Trajectory Equation

Instead of differentiating with respect to time as in Eqs. (6) and (7), one can differentiate with respect to θ by using the following relations:

$$\dot{r} = \dot{\theta}\,\frac{\mathrm{d}r}{\mathrm{d}\theta} \tag{16}$$

$$\ddot{r} = \ddot{\theta}\,\frac{\mathrm{d}r}{\mathrm{d}\theta} + \dot{\theta}^2\,\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} \tag{17}$$

By substituting Eqs. (16) and (17) in Eq. (6) for the radial acceleration, one gets the second-order differential equation

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} - \frac{2}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 - \left(1 - \frac{g}{\dot{\theta}^2}\right) r = 0 \tag{18}$$

By substituting Eqs. (14) and (15) into Eq. (18), one gets

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} - \left(1 - \frac{\dot{\psi}}{\dot{\theta}}\right)\left(\frac{1}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{\dot{\psi}}{\dot{\theta}} r = 0 \tag{19}$$

When the proportional navigation law $k = \dot{\psi}/\dot{\theta}$, where k is a constant, is used, Eq. (19) integrates to

$$r^k = C|\sin(k\theta + b)| \tag{20}$$

which is similar to the result derived by Lu[6] for the case of a nonmoving target.

On the other hand, if we assume that the radial acceleration in Eq. (6) is zero, then Eq. (18) reduces to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} + \frac{\ddot{\theta}}{\dot{\theta}^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} - r = 0 \tag{21}$$

In the case where the target velocity v_t is constant and horizontal, it is easily shown that $\ddot{\theta}/\dot{\theta}^2 = 2\cot\theta$, which is the case treated by Jalali-Naini and Esfahanian.[7] The solution in this case is given by $r\sin\theta = A_1(\theta - \theta_0)$, and one can easily derive for the transverse acceleration [Eq. (7)] $a_\theta = \dot{l}/r = (2A_1 v_t^2/h)\sin^3\theta$, which is Eq. (22) of Ref. 7 (h is the constant vertical distance of the target from the x axis).

IV. Conclusions

This study shows that some scattered published results in the field of guidance theory can be derived directly from a general differential equation derived from classical mechanics. This approach offers the possibility of allowing a unified approach for various methods used to model the trajectory of a homing missile.

References

[1] Shoucri, R. M., "Elliptical Orbit with Variable Angular Momentum," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 5, 1995, pp. 1213–1215.

[2] Shoucri, R. M., "Constant Keplerian Orbit with Non-Central Force Field," *International Journal of Engineering Education*, Vol. 12, No. 4, 1996, pp. 305–308.

[3] Shoucri, R. M., "Note on Kepler's Problem with Variable Angular Momentum," *Celestial Mechanics and Dynamical Astronomy*, Vol. 63, No. 2, 1996, pp. 75–85.

[4] Shoucri, R. M., "Elliptical Orbit with Non-Central Force Field in Celestial Mechanics," *Celestial Mechanics and Dynamical Astronomy*, Vol. 65, No. 4, 1997, pp. 373–388.

[5] Yang, C.-D., Yeh, F.-B., and Chen, J.-H., "The Closed-Form Solution of Generalized Proportional Navigation," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 2, 1987, pp. 216–218.

[6] Lu, P., "Intercept of Nonmoving Targets at Arbitrary Time-Varying Velocity," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 176–178.

[7] Jalali-Naini, S. H., and Esfahanian, V., "Closed-Form Solution of Line-of-Sight Trajectory for Nonmaneuvering Targets," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 2, 2000, pp. 365, 366.

Output-Rate Weighted Optimal Control of Aeroelastic Systems

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Introduction

THE optimal linear quadratic regulator (LQR) problem[1] is the backbone of many modern optimal, robust control design methods, such as the linear quadratic Gaussian procedure with loop-transfer recovery (LQG/LTR)[2] and the H_2/H_∞ control. The output-weighted LQR problem (referred to as LQRY) minimizes an objective function containing the quadratic form of the measured output as an integrand. However, in several applications it may be more desirable to minimize the time rate of change of the output, rather than the output itself. Examples of such active control applications are vibration reduction of flexible structures, flutter suppression, gust and maneuver load alleviation of aircraft, and ride-quality augmentation of any vehicle. In these cases, the measured output is usually the normal acceleration, although it is necessary from consideration of passenger/crew comfort (as well as issues such as weapons aiming and delivery) to have an optimal controller that minimizes the roughness of the motion, which can be defined as the time rate of change of normal acceleration. This mechanical analogy can be extended to other physical systems, where sensor limitations prevent the measurement of time rate of change of an available signal, and even to economic models. Also, it is well known that certain jerky nonlinear motions, if uncorrected, can lead to chaos.[3] Optimal control of such motions may require output-rate weighted (ORW) objective functions.

ORW Optimal Control

Consider a linear time-invariant system described by the following state-space equation:

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{F}\boldsymbol{v} \tag{1}$$

where $\boldsymbol{X}$ is the state vector, $\boldsymbol{u}$ is the input vector, and $\boldsymbol{v}$ is the process noise vector. Then the output vector $\boldsymbol{y}$ is given by

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{X} + \boldsymbol{D}\boldsymbol{u} + \boldsymbol{w} \tag{2}$$

where $\boldsymbol{w}$ is the measurement noise vector. The regulator design problem is finding an optimal feedback gain matrix $\boldsymbol{K}$, which obeys the control law

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{Z} \tag{3}$$

where $\boldsymbol{Z}$ is the estimated state vector, such that the following objective function is minimized:

$$J = \left(\frac{1}{2}\right)\int_0^\infty \left[\left(\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}\right)^T \boldsymbol{Q}\left(\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}\right) + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\right]\mathrm{d}t \tag{4}$$

Equation (4) can be rewritten as

$$J = \left(\frac{1}{2}\right)\int_0^\infty (\boldsymbol{X}'\bar{\boldsymbol{Q}}\boldsymbol{X} + \boldsymbol{X}'\boldsymbol{S}'\boldsymbol{U} + \boldsymbol{U}'\boldsymbol{S}\boldsymbol{X} + \boldsymbol{U}'\bar{\boldsymbol{R}}\boldsymbol{U})\,\mathrm{d}t \tag{5}$$

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